Martingale methods in stochastic differential equations

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Stieltjes Day, VU Amsterdam (March 5, 2009)

S. Varadhan, Abel Prize recipient 2007

The concentration of diffusing particles in $\mathbb{R}^d$ is described by the forward Kolmogorov equation

$$D_t v(t,x) = \frac{1}{2}\sum_{i,j=1}^{d} D_i D_j (a_{ij}(x) v(t,x)) - \sum_{i=1}^{d} D_i (b_i(x) v(t,x)).$$

Here

$$a : \mathbb{R}^d \to \mathbb{S}^d \quad (d \times d \text{ real symmetric matrices})$$
$$b : \mathbb{R}^d \to \mathbb{R}^d$$

are the diffusion and drift parameters.

To solve this equation, a_{ij} and b_j must be sufficiently smooth.

I: PDE approach

Consider the adjoint backward Kolmogorov equation

$$D_t u(t,x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x) D_i D_j u(t,x) + \sum_{i=1}^{d} b_i(x) D_i u(t,x).$$

Theorem. *Suppose a and b are bounded and Hölder continuous, and a satisfies the nondegeneracy condition*

$$\langle a(x)y, y\rangle \geqslant \lambda |y|^2 \quad (y \in \mathbb{R}^d).$$

Then the above problem admits a fundamental solution p such that for any terminal value $u(t,x) = f(x)$ *with* $f \in C_{\mathrm{b}}(\mathbb{R}^d)$*, the solution is given by*

$$u(s,x) = \int_{\mathbb{R}^d} p_{t-s}(x,y) f(y)\, dy \quad (0 \leqslant s < t).$$

Define the operators $P_t : B_{\mathrm{b}}(\mathbb{R}^d) \to B_{\mathrm{b}}(\mathbb{R}^d)$ by

$$P_t f(x) = \int_{\mathbb{R}^d} p_t(x,y) f(y)\, dy.$$

Then P_t is a Feller semigroup on $C_0(\mathbb{R}^d)$, i.e.,

- $P_0 = I$, $P_t \circ P_s = P_{t+s}$
- $0 \leqslant f \leqslant 1 \implies 0 \leqslant P_t f \leqslant 1$
- $f \in C_0(\mathbb{R}^d) \implies P_t f \in C_0(\mathbb{R}^d)$
- $\lim_{t \downarrow 0} P_t f = f$ for all $f \in C_0(\mathbb{R}^d)$

P_t is conservative, i.e., $P_t 1 = 1$.
Its generator extends the operator L, i.e., $C_{\mathrm{c}}^2(\mathbb{R}^d) \subseteq \mathscr{D}(L)$ and

$$P_t f = f + \int_0^t P_r L f\, dr \quad (f \in C_{\mathrm{c}}^2(\mathbb{R}^d)).$$

Let P_t be a conservative Feller semigroup with local generator L:

$$f \equiv 0 \quad \text{on } U \quad \Longrightarrow \quad Lf \equiv 0 \quad \text{on } U.$$

For each $x \in \mathbb{R}^d$ there exists a unique probability measure $\mathbb{P}^x$ on

$$\Omega := C([0,\infty);\mathbb{R}^d)$$

such that the coordinate process

$$X_t(\omega) := \omega_t \quad (\omega \in \Omega)$$

is a Markov process starting at x with transition semigroup P_t, i.e., $\mathbb{P}^x$-a.s. one has $X_0 = x$ and

$$\mathbb{E}^x(f(X_{t+s})|\mathscr{F}_s) = P_t f(X_s).$$

For all $x \in X$ and $f \in C_c^2(\mathbb{R}^d)$,

$$M_t^x := f(X_t) - \int_0^t Lf(X_r)\,dr$$

is a martingale with respect to $\mathbb{P}^x$, i.e.,

$$\mathbb{E}^x(M_t^x | \mathscr{F}_s) = M_s^x \quad (t \geqslant s).$$

Indeed,

$$\begin{aligned}
&\mathbb{E}^x(f(X_t) - f(X_s) | \mathscr{F}_s) - \int_s^t \mathbb{E}^x(Lf(X_r) | \mathscr{F}_s)\,dr \\
&= P_{t-s}f(X_s) - f(X_s) - \int_s^t P_{r-s}Lf(X_s)\,dr \\
&= P_{t-s}f(X_s) - f(X_s) - \int_0^{t-s} P_r Lf(X_s)\,dr \\
&= 0.
\end{aligned}$$

II: SDE approach

Let B_t be a Brownian motion in $\mathbb{R}^d$.

Suppose that

$$a(x) = \sigma(x)\sigma^*(x)$$

and consider the stochastic differential equation

$$\begin{cases} dU_t = b(U_t)\,dt + \sigma(U_t)\,dB_t \\ U_0 = x \end{cases}$$

A solution is a continuous adapted process U_t^x in $\mathbb{R}^d$ such that

$$U_t^x = x + \int_0^t b(U_r^x)\,dr + \int_0^t \sigma(U_r^x)\,dB_r$$

Theorem. (Itô) *Assume*

- $b : \mathbb{R}^d \to \mathbb{R}^d$ *is bounded and Lipschitz continuous*
- $\sigma : \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d)$ *is bounded and Lipschitz continuous*
- $x \in \mathbb{R}^d$.

Then the stochastic differential equation

$$\begin{cases} dU_t = b(U_t)\, dt + \sigma(U_t)\, dB_t \\ U_0 = x \end{cases}$$

admits a unique continuous solution U_t^x.

Proof: Picard iteration.

By Itô's formula, for $f \in C_c^2(\mathbb{R}^d)$ one has

$$df(U_t^x) = Df(U_t^x)\, dU_t^x + \frac{1}{2}\sum_{i,j=1}^{d} D^2 f(U_t^x)\, d[U^x]_t$$
$$= Df(U_t^x)\sigma(U_t^x)\, dB_t + Lf(X_t)\, dt$$

since $[U^x]_t = \sigma(U_t^x)\sigma^*(U_t^x)\, dt = a(U_t^x)\, dt$.

It follows that

$$f(U_t^x) - \int_0^t Lf(U_r^x)\, dr = f(x) + \int_0^t Df(U_r^x)\sigma(U_r^x)\, dB_r$$

is a martingale.

III: Martingale approach

Let

$$Lf = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij} D_i D_j f + \sum_{j=1}^{d} b_j D_j f$$

for $f \in C_c^2(\mathbb{R}^d)$, with $a : \mathbb{R}^d \to \mathbb{S}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ as before.

A probability measure $\mathbb{P}$ on Ω solves the martingale problem for L starting at x if

- $X_0 = x \quad \mathbb{P}$-a.s.
- For all $f \in C_c^2(\mathbb{R}^d)$

$$f(X_t) - \int_0^t Lf(X_r)\, dr$$

is a martingale with respect to $\mathbb{P}$.

The martingale problem is well-posed if for every $x \in \mathbb{R}^d$ there is a unique solution $\mathbb{P}^x$.

Example. Take $a_{ij} = \delta_{ij}$ and $b_j = 0$, so

$$L = \tfrac{1}{2}\Delta.$$

If B_t^x is a Brownian motion starting at x, then by Itô's formula, for $f \in C_c^2(\mathbb{R}^d)$

$$f(B_t^x) - \int_0^t \tfrac{1}{2}\Delta f(B_r^x)\,dr = f(x) + \int_0^t Df(B_r^x)\,dB_r$$

is a martingale.

Hence the law of B_t^x solves the martingale problem.

Conversely, suppose that $\mathbb{P}^x$ solves the martingale problem for $L = \frac{1}{2}\Delta$.

By a stopping time argument, for all $f \in C^2(\mathbb{R}^d)$

$$f(X_t) - \int_0^t \tfrac{1}{2}\Delta f(X_r)\, dr$$

is a continuous local martingale.

Taking $f_j(x) = x_j$ and $g_j(x) = x_j^2$,

$$X_t^j \quad \text{and} \quad (X_t^j)^2 - t$$

are continuous local martingales with respect to $\mathbb{P}^x$.

Theorem (Lévy) *With respect to* $\mathbb{P}^x$, X_t *is a Brownian motion starting at* x.

Theorem. (Stroock-Varadhan) *Consider*

$$Lf = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij} D_i D_j f + \sum_{j=1}^{d} b_j D_j f\,.$$

Assume

- $a : \mathbb{R}^d \to \mathbb{S}^d$ *is bounded and continuous and satisfies*

$$\langle a(x)y, y\rangle \geqslant \lambda |y|^2$$

- $b : \mathbb{R}^d \to \mathbb{R}^d$ *is bounded and measurable.*

Then:

- *The martingale problem for* L *is well-posed.*
- *The process* X_t *has the strong Markov property with respect to the measures* $\mathbb{P}^x$.
- *If* b *is continuous, there exists a unique Feller semigroup* P_t *on* $C_0(\mathbb{R}^d)$ *whose generator extends* L.

Sketch of the proof

Existence:

- Discretization
- Weak compactness of families of probability measures on Ω.

Uniqueness:

- By a Cameron-Martin-Girsanov transformation:

  WLOG $b \equiv 0$.
- By localisation arguments:

  WLOG $|a_{ij}(x) - \delta_{ij}| < \eta$.

Sketch of the proof

To deal with this case, note that

$$\mathbb{P}^x = \widetilde{\mathbb{P}}^x \iff R^x(\lambda)f = \widetilde{R}^x(\lambda)f \quad (\lambda > 0,\ f \in C_c^2(\mathbb{R}^d))$$

where

$$R^x(\lambda)f = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t)\, dt.$$

By the martingale property and integration by parts,

$$R^x(\lambda)(\lambda - L)f = f(x).$$

Formally,

$$\underbrace{(\lambda - L)^{-1}}_{R(\lambda)} = \underbrace{(\lambda - \tfrac{1}{2}\Delta)^{-1}}_{R_{BM}(\lambda)} \left(I - \underbrace{(L - \tfrac{1}{2}\Delta)}_{\frac{1}{2}(a-\delta)} \underbrace{(\lambda - \tfrac{1}{2}\Delta)^{-1}}_{R_{BM}(\lambda)} \right)^{-1}$$

where, for Brownian motion,

$$R_{BM}^x(\lambda)f = \mathbb{E}_{BM}^x \int_0^\infty e^{-\lambda t} f(X_t)\, dt.$$

Rewriting, this gives

$$R^x_{BM}(\lambda)f = R^x(\lambda)f - R^x(\lambda)\Big(\frac{1}{2}\sum_{i,j=1}^{d}(a_{ij}-\delta_{ij})D_iD_j\Big)U_\lambda f\,.$$

Subtracting these identities for $\mathbb{P}^x$ and $\widetilde{\mathbb{P}}^x$ gives

$$\|R^x(\lambda)-\widetilde{R}^x(\lambda)\| \leqslant \frac{\eta}{2}\Big(\max_{1\leqslant i,j\leqslant d}\|D_iD_jU_\lambda\|\Big)\|R^x(\lambda)-\widetilde{R}^x(\lambda)\|.$$

For $p > d/2$, $R^x(\lambda)$ and $\widetilde{R}^x(\lambda)$ are bounded on L^p.

L^p-Boundedness of Riesz transforms $\rightarrow$ L^p-boundedness of $D_iD_jU_\lambda$

For $\eta > 0$ small this gives

$$R^x(\lambda) = \widetilde{R}^x(\lambda).$$

Invariance principles

Let $X_n : A \to \mathbb{R}^d$ be i.i.d. standard normal random variables.

Given $x \in \mathbb{R}^d$ and $h > 0$, define $\Phi^{x,h} : A \to \Omega$ by

$$\Phi^{x,h} = x + \sqrt{h}\Big(\sum_{n=1}^{\lfloor t/h \rfloor} X_n + \big(t - h\lfloor t/h \rfloor\big) X_{\lfloor t/h \rfloor + 1}\Big).$$

Let $P^{x,h}$ be its law.

Theorem. (Donsker) $\lim_{h\downarrow 0,\, y\to x} \mathbb{P}^{y,h} = \mathbb{P}^x_{BM}$ *weakly.*

Stroock-Varadhan theory implies, more generally, convergence of Markov chains to diffusions.

References

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